Riemannian Geometry | Week 5

<u>Recall</u>: Connection ∇ on M: $\nabla: \Gamma(M) \times \Gamma(M) \longrightarrow \Gamma(M)$,

$$\underset{\substack{\nearrow \\ C^\infty(M)\text{-linear}}}{\nabla_X} \underset{\uparrow \mathbb{R}\text{-linear}}{Y}$$

, Leibnitz rule w.r.t 2^{nd} argument

It is local: If $U \subseteq M$ is open and $p \in U$:

$\nabla_X Y|_p$ depends only on $X|_p$ and $Y|_U$

Given local coordinates $(x^1, \dots, x^n)$: Christoffel symbols (associated to the coordinate system):

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

So $(\nabla_X Y)^k = X^i \frac{\partial}{\partial x^i} Y^k + \Gamma^k_{ij} X^i Y^j$

In the exercises: ∇_X extends to <u>tensor fields</u>, so that

$$\nabla_X (f \otimes g) = (\nabla_X f) \otimes g + f \otimes (\nabla_X g), \qquad \nabla_X (\mathrm{tr} f) = \mathrm{tr}(\nabla_X f).$$

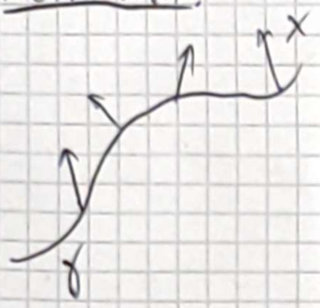
- So for 1-forms:

$$(\nabla_X \omega)_k = X^i \frac{\partial}{\partial x^i} \omega_k - \Gamma^a_{ik} X^i \omega_a$$

- For (1,1) tensor fields:

$$(\nabla_X T)^a_b = X^i \frac{\partial}{\partial x^i} T^a_b + \Gamma^a_{ij} X^i T^j_b - \Gamma^m_{ib} T^a_m$$

<u>Definition:</u>



A vector field X along a curve γ is a smooth map

$$t \longrightarrow X_t \in T_{\gamma(t)} M$$

<u>Note:</u>

If γ self-intersects ($\gamma(t_1) = \gamma(t_2)$): X_{t_1} doesn't have to be the same as X_{t_2} (so not unique value at all points of M belonging to the image of γ).

- In this case: X doesn't extend to a vector field on M.

If γ is <u>embedded</u> curve: X always extends (non-uniquely) to an element of $\Gamma(M)$.

- Locally (i.e. restricting the domain of γ): Any curve is embedded

<u>Covariant derivative of X along γ:</u> Defined uniquely, independently of the (local) extension of X around $\gamma(t)$.

<u>Sketch of proof:</u> Extend X to a vector field in a neighborhood of $\gamma(t)$ in M. • Then, in local coordinates around $\gamma(t)$:

$$\left(\nabla_{\dot\gamma} X\big|_{\gamma(t)}\right)^k = \dot\gamma^i \frac{\partial}{\partial x^i} X^k\Big|_{\gamma(t)} + \Gamma^k_{ij}(\gamma(t))\, \dot\gamma^i(t)\, X^j\big|_{\gamma(t)}$$

$$= \frac{d}{dt} X^k_t + \Gamma^k_{ij}(\gamma(t)) \cdot \dot\gamma^i(t)\, X^j_t$$

← Depends only on the values along γ

☐

Levi-Civita connection:

On a general manifold: An infinite family of connections ($\nabla - \bar{\nabla}$ is always a $(1,2)$-tensor field)

If M is equiped with a Riemannian metric: A unique natural connection:

Theorem: Let (M, g) be a Riem. manifold. There exists a unique ∇ (called the Levi-Civita connection) such that:

① ∇ is torsion-free $\left(\Gamma^k_{ij} = \Gamma^k_{ji}\right)$ $\leftarrow$ If true in one coordinate system, it's true for all, since the torsion is a tensor

② ∇ respects the metric

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

$$(\Leftrightarrow \nabla_X g = 0)$$

Example: On $(\mathbb{R}^n, g_E)$, the flat connection is the Levi-civita connection.

Proof: • Uniqueness: It suffices to prove that in any local coordinate system, conditions ① & ② uniquely fix the Christoffel symbols.

In $(x^1, \dots, x^n)$:

$$\partial_i g_{jk} = \partial_i (g(\partial_j, \partial_k)) \overset{②}{=} g(\nabla_{\partial_i} \partial_j, \partial_k) + g(\partial_j, \nabla_{\partial_i} \partial_k)$$

$$= g(\Gamma^a_{ij} \partial_a, \partial_k) + g(\partial_j, \Gamma^a_{ik} \partial_a) \Rightarrow$$

$$\Rightarrow \quad \partial_i g_{jk} = \Gamma^a_{ij} g_{ak} + \Gamma^a_{ik} g_{aj} \quad (a)$$

$$(i \leftrightarrow k) \quad \partial_k g_{ij} = \Gamma^a_{kj} g_{ai} + \Gamma^a_{ki} g_{aj} \quad (b)$$

$(i \leftrightarrow j)$ $\partial_j g_{ik} = \Gamma^a$ [illegible] $+ \Gamma^a$ [illegible] (c)

So ⓐ + ⓑ − ⓒ : Using also that $\Gamma_{ab}^{c} = \Gamma_{ba}^{c}$

$$\partial_i g_{jk} + \partial_k g_{ji} - \partial_j g_{ik} = 2\Gamma^{a}_{ik} g_{aj}$$

$$\Rightarrow \boxed{\Gamma^{\gamma}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}\left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\delta\alpha} - \partial_\delta g_{\alpha\beta} \right).}$$

Where $[g^{ij}]$: The inverse matrix of $[g_{ij}]$.

- Existence: Let use the shorthand notation $\langle \cdot,\cdot \rangle := g(\cdot,\cdot)$

In order to define $\nabla_X Y$ $\forall X, Y \in \Gamma(M)$, it suffices to define $\langle \nabla_X Y, Z \rangle$ $\forall X, Y, Z \in \Gamma(M)$.

ⓐ $X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

Torsion free: $\nabla_Y X = \nabla_X Y + [Y,X]$

ⓑ $Y\langle X, Z\rangle = \langle \nabla_Y X, Z\rangle + \langle X, \nabla_Y Z\rangle =$

$= \langle \nabla_X Y, Z\rangle + \langle [Y,X], Z\rangle + \langle X, \nabla_Y Z\rangle$

ⓒ $Z\langle X, Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \overset{\text{Torsion free}}{=}$

$= \langle \nabla_X Z, Y\rangle + \langle [Z,X], Y\rangle + \langle X, \nabla_Y Z\rangle + \langle X, [Z,Y]\rangle$

ⓐ + ⓑ − ⓒ:

$$X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle = 2\langle \nabla_X Y, Z\rangle + \langle [Y,X], Z\rangle - \langle [Z,X], Y\rangle - \langle [Z,Y], X\rangle$$

$$\Rightarrow \langle \nabla_X Y, Z\rangle = \frac{1}{2}\Big(X(\langle Y,Z\rangle) + Y(\langle X,Z\rangle) - Z(\langle X,Y\rangle) + \langle [X,Y], Z\rangle + \langle [Z,X], Y\rangle - \langle [Y,Z], X\rangle \Big)$$

<u>Koszul's formula.</u> ◻

L-C connection: The <u>canonical</u> connection on (M,g)

From now on: When referring to the connection of a Riemannian manifold, we will always mean the L-C connection).

<u>Note:</u> If $F:(M,g) \longrightarrow (N,h)$ is a (local) isometry, it "preserves" the L-C connection, i.e.

If $X, Y \in \Gamma(M)$:

$$dF(\nabla^{(g)}_X Y) = \nabla^{(h)}_{dF(X)} dF(Y).$$

<u>Proof</u>: All the terms in the Koszul formula are "respected" by dF:
- $\langle dF(v), dF(w)\rangle_h = \langle v, w\rangle_g$ (F: isometry)
- $[dF(v), dF(w)] = dF([v,w])$ for any smooth map.

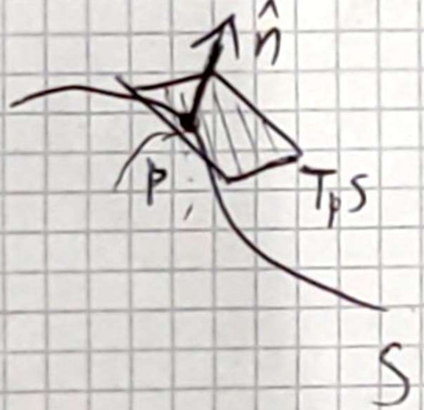
◻

Important example (We will discuss this in more detail later)

Let $S \subseteq \mathbb{R}^n$ be a hypersurface (e.g. the graph of a function).

For the induced metric g on S from g_E:

The L-C connection ∇ is the projection of the flat connection on $\mathbb{R}^n$ on $TS \subseteq T\mathbb{R}^n$

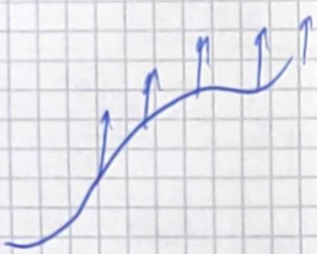


$$(\nabla_X Y)|_p = \pi^T(D_X Y|_p)$$

projection on T_pS

Parallel transport:

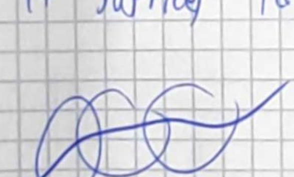
Def: Let $\gamma : (0,1) \to M$ be smooth, and ∇ a connection on M, $X \in \Gamma_\gamma$. We will say that X is parallel transported along γ if $\nabla_{\dot\gamma} X = 0$



Thm: $\forall\ \xi \in T_{\gamma(0)} M$: $\exists!\ X \in \Gamma_\gamma$: 1) $X|_{\gamma(0)} = \xi$
2) $\nabla_{\dot\gamma} X = 0$

Parallel transport of ξ : $X = P_\gamma \xi$

Proof: It suffices to do the construction in a local coordinate chart on $\gamma|_{(t_i, t_{i+1})}$



$$(\nabla_{\dot\gamma} X)^k = \frac{d}{dt} X^k + \Gamma^k_{ij} \dot\gamma^i X^j = 0$$

↑ Linear ODE : $\exists$ unique solution

$\blacksquare$

Prop: - $P_{\gamma(t)} : T_{\gamma(0)} M \to T_{\gamma(t)} M$ linear bijection!

- If ∇ L-C connection of (M,g) : P_γ isometry!!

Geodesic : Self-parallel curve

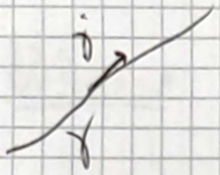
$\nabla_{\dot\gamma}\dot\gamma = 0$ In local coordinates:

$$\boxed{\ddot\gamma^k + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0}$$

Existence and uniqueness

Thm : $\forall p \in M$, $v \in T_pM$: $\exists !$ maximal $\gamma : (-T_-, T_+) \to M$ $\quad T_\pm \in \mathbb{R}$

such that $\gamma(0) = p$, $\dot\gamma(0) = v$, $\nabla_{\dot\gamma}\dot\gamma = 0$



$\nabla_{\dot\gamma}\dot\gamma = 0$ In local coordinates:

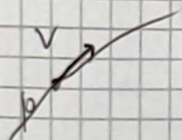
$t \longmapsto (\gamma^1(t), \dots, \gamma^n(t))$

$$\ddot\gamma^k(t) + \Gamma^k_{ij}\dot\gamma^i\dot\gamma^j = 0$$

From ODE theory:

$\forall p \in M$, $\forall v \in T_pM$: $\exists!$ maximal geodesic $\gamma : (-T_-, T_+) \to M$, $(-T_-, T_+) \subseteq \mathbb{R}$

such that $\gamma(0) = p$, $\dot\gamma(0) = v$



Denote it with $\gamma_{p,v}$: In local coordinates: $\gamma^k_{p,v}(t)$ depends smoothly on p, v.

And: If $\left.\begin{array}{l}\gamma_1(0) = \gamma_2(0)\\ \dot\gamma_1(0) = \dot\gamma_2(0)\end{array}\right\}$ $\gamma_1 = \gamma_2$ on their common domain of definition

On flat space: Geodesics (aka straight lines) minimize the distance between points.

On a general Riemannian manifold: Not necessarily true (example: sphere). But geodesics are still stationary points of the length functional: